

# Bounds for Hochschild cohomology of block algebras

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## Abstract

We show that for any block algebra  $B$  of a finite group over an algebraically closed field of prime characteristic  $p$  the dimension of  $HH^n(B)$  is bounded by a function depending only on the nonnegative integer  $n$  and the defect of  $B$ . The proof uses in particular a theorem of Brauer and Feit which implies the result for  $n = 0$ .

Let  $p$  be a prime and  $k$  an algebraically closed field of characteristic  $p$ . Let  $G$  be a finite group and  $B$  a block algebra of  $kG$ ; that is,  $B$  is an indecomposable direct factor of  $kG$  as a  $k$ -algebra. A *defect group* of  $B$  is a minimal subgroup  $P$  of  $G$  such that  $B$  is isomorphic to a direct summand of  $B \otimes_{kP} B$  as a  $B$ - $B$ -bimodule. The defect groups of  $B$  form a  $G$ -conjugacy class of  $p$ -subgroups of  $G$ , and the *defect* of  $B$  is the integer  $d(B)$  such that  $p^{d(B)}$  is the order of the defect groups of  $B$ . The *weak Donovan conjecture* states that the Cartan invariants of  $B$  are bounded by a function depending only on the defect  $d(B)$  of  $B$ . As a consequence of a theorem of Brauer and Feit [3], the number of isomorphism classes of simple  $B$ -modules is bounded by a function depending only on  $d(B)$ . Thus the weak Donovan conjecture would imply that the dimension of a basic algebra of  $B$  is bounded by a function depending on  $d(B)$ . This in turn would imply that the dimension of the term in any fixed degree  $n$  of the Hochschild complex of a basic algebra of  $B$  is bounded by a function depending on  $n$  and  $d(B)$ ; since Hochschild cohomology is invariant under Morita equivalences, we would thus get that the dimension of  $HH^n(B)$  is bounded by a function depending on  $n$  and  $d(B)$ . The purpose of this note is to show that this consequence of the weak Donovan conjecture does indeed hold.

**Theorem 1.** *There is a function  $f : \mathbb{N}_0 \times \mathbb{N}_0 \rightarrow \mathbb{N}_0$  such that for any integer  $n \geq 0$ , any finite group  $G$  and any block algebra  $B$  of  $kG$  with defect  $d$  we have*

$$\dim_k(HH^n(B)) \leq f(n, d)$$

For  $n = 0$  this follows from the aforementioned theorem of Brauer and Feit [3], since  $HH^0(B) \cong Z(B)$ . Using Tate duality, the theorem above extends to Tate cohomology for negative  $n$ . A result of Külshammer and Robinson [6, Theorem 1] implies that it suffices to show theorem 1 for finite groups with a non-trivial normal  $p$ -subgroup. We follow a slightly different strategy in the proof below, reducing the problem directly to finite groups with a non-trivial central  $p$ -subgroup.

**Remark 2.** We make no effort to construct a best possible bound; we define the function  $f$  in theorem 1 inductively as follows: we set  $f(0, 0) = 1$ ,  $f(n, 0) = 0$  for  $n > 0$ ; for all  $d > 0$ ,  $f(0, d)$  is

the largest integer less or equal to the bound  $\frac{1}{4}p^{2d} + 1$  given in the Brauer-Feit theorem, and for  $n > 0, d > 0$  we set

$$f(n, d) = p \cdot c(d) \cdot \sum_{i=0}^n f(i, d-1)$$

where  $c(d)$  is the maximum of the numbers of subgroups in any finite group of order  $p^d$ .

Let  $G$  be a finite group and  $U$  a  $kG$ -module. We denote as usual by  $U^G$  the subspace of  $G$ -fixed points in  $U$ . If  $H$  is a subgroup of  $G$  then  $U^G \subseteq U^H$ , and there is a *trace map*  $\text{tr}_H^G : U^H \rightarrow U^G$  sending  $u \in U^H$  to  $\sum_{x \in [G/H]} xu$ , where  $[G/H]$  is a set of representatives of the  $H$ -cosets in  $G$ ; one checks that this map is independent of the choice of  $[G/H]$  and that its image, denoted  $U_H^G$ , is contained in  $U^G$ . For  $Q$  a  $p$ -subgroup of  $G$ , we denote the Brauer construction of  $U$  with respect to  $Q$  by  $U(Q) = U^Q / \sum_{R: R < Q} U_R^Q$  and by  $\text{Br}_Q^U : U^Q \rightarrow U(Q)$  the canonical surjection, called *Brauer homomorphism*. A block algebra  $B$  of  $kG$  can be viewed as an indecomposable  $k(G \times G)$ -module, with  $(x, y) \in G \times G$  acting by left multiplication with  $x$  and right multiplication with  $y^{-1}$ . For  $H$  a subgroup of  $G$ , we denote by  $\Delta H$  the ‘diagonal’ subgroup  $\Delta H = \{(h, h) \mid h \in H\}$  in  $G \times G$ . In particular, the action of  $\Delta G$  on  $B$  can be identified with the conjugation action of  $G$  on  $B$ . The Brauer construction applied to  $B$  with respect to  $\Delta Q$  is canonically isomorphic to  $kC_G(Q)c$ , where  $Q$  is a  $p$ -subgroup of  $G$  and  $c = \text{Br}_{\Delta Q}(1_B)$ . A *B-Brauer pair* is a pair  $(Q, e)$  consisting of a  $p$ -subgroup  $Q$  of  $G$  and of a block idempotent  $e$  of  $kC_G(Q)$  satisfying  $e\text{Br}_Q(1_B) = e$ . The set of  $B$ -Brauer pairs is a  $G$ -poset in which the maximal pairs are all conjugate. The maximal  $B$ -Brauer pairs are exactly the  $B$ -Brauer pairs  $(Q, e)$  for which  $Q$  is a defect group of  $B$ . See [2] and [8, §11, §40] for details. In what follows we use without further comment the canonical graded isomorphism  $HH^*(B) \cong H^*(\Delta G; B)$ ; see [7, (3.2)]. The following result is certainly well-known but not always stated in exactly the form we need it; we therefore give a proof for the convenience of the reader.

**Proposition 3.** *Let  $G$  be a finite group,  $B$  be a block algebra of  $kG$  and  $Q$  a  $p$ -subgroup of  $G$ . Set  $b = 1_B$  and  $c = \text{Br}_Q(b)$ . Suppose that  $c \neq 0$  and set  $B_Q = kC_G(Q)cb$ . Then we have a direct sum decomposition of  $kN_{G \times G}(\Delta Q)$ -modules*

$$\text{Res}_{N_{G \times G}(\Delta Q)}^{G \times G}(B) = B_Q \oplus C_Q$$

*such that multiplication by  $b$  is an isomorphism of  $kN_{G \times G}(\Delta Q)$ -modules  $kC_G(Q)c \cong B_Q$  and such that  $C_Q(\Delta Q) = \{0\}$ .*

The proof we present here uses the following well-known lemma, which is a special case of expressing relative projectivity in terms of the splitting of adjunction maps (the general theme behind this is developed in [4], [5], for instance).

**Lemma 4.** *Let  $\alpha : B \rightarrow A$  be a homomorphism of  $k$ -algebras. Suppose that  $B$  is isomorphic to a direct summand of  $A$  as a  $B$ - $B$ -bimodule. Then  $\alpha$  is injective and  $\text{Im}(\alpha)$  is a direct summand of  $A$  as a  $B$ - $B$ -bimodule.*

*Proof.* The left or right action of an element  $b \in B$  on  $A$  is given by left or right multiplication with  $\alpha(b)$ . Let  $\iota : B \rightarrow A$  and  $\pi : A \rightarrow B$  be  $B$ - $B$ -bimodule homomorphisms satisfying  $\pi \circ \iota = \text{Id}_B$ . Then  $\iota(1_B)$  commutes with  $\text{Im}(\alpha)$ , the map  $\beta$  sending  $a \in A$  to  $a\iota(1_B)$  is an  $A$ - $B$ -bimodule

endomorphism of  $A$ , and we have  $\beta(\alpha(b)) = \alpha(b)\iota(1_A) = \iota(b)$ , hence  $\beta \circ \alpha = \iota$ . Thus  $\pi \circ \beta \circ \alpha = \text{Id}_B$ , which shows that as a  $B$ - $B$ -bimodule homomorphism,  $\alpha$  is split injective with  $\pi \circ \beta$  as a retraction.  $\square$

*Proof of Proposition 3.* For any block of  $kN_G(Q)$  which appears in  $kN_G(Q)c$ , the block  $B$  of  $kG$  is the corresponding ‘induced’ block. By [1, §14, Lemma 1],  $kN_G(Q)c$  is isomorphic to a direct summand of  $B$  as a  $kN_G(Q)$ - $kN_G(Q)$ -bimodule, and thus of  $cBc$ , as a  $kN_G(Q)c$ - $kN_G(Q)c$ -bimodule. By lemma 4, multiplication by  $b$  induces an algebra homomorphism  $kN_G(Q)c \rightarrow cBc$  which is split injective as a homomorphism of  $kN_G(Q)c$ - $kN_G(Q)c$ -bimodules. Since  $kC_G(Q)c$  is a direct summand of  $kN_G(Q)c$  as an  $N_{G \times G}(\Delta Q)$ -module we get that  $kC_G(Q)c \cong B_Q$  and that  $B_Q$  is a direct summand of  $B$  as an  $N_{G \times G}(\Delta Q)$ -module. Moreover,  $B(\Delta Q) \cong B_Q$ , and hence any complement  $C_Q$  of  $B_Q$  in  $B$ , as an  $N_{G \times G}(\Delta Q)$ -module, satisfies  $C_Q(\Delta Q) = \{0\}$ .  $\square$

We will make use of the following well-known fact on transfer in cohomology (we include a short proof for the convenience of the reader).

**Lemma 5.** *Let  $G$  be a finite group,  $H$  a subgroup of  $G$  and  $V$  a  $kH$ -module. Let  $U$  be a direct summand of  $\text{Ind}_H^G(V)$ . Then  $H^*(G; U) = \text{tr}_H^G(H^*(H; \text{Res}_H^G(U)))$ .*

*Proof.* By Higman’s criterion there is a  $kH$ -endomorphism  $\varphi$  of  $U$  such that  $\text{Id}_U = \text{tr}_H^G(\varphi)$ . Let  $n \geq 0$  and let  $\zeta : \Omega^n(k) \rightarrow U$  be a  $kG$ -homomorphism, representing an element in  $H^n(G; U)$ . Then  $\zeta = \text{Id}_U \circ \zeta = \text{tr}_H^G(\varphi \circ \zeta)$ , whence the result.  $\square$

This is applied in the following situation:

**Lemma 6.** *Let  $G$  be a finite group,  $B$  a block algebra of  $kG$  and  $P$  a defect group of  $B$ . We have  $H^*(\Delta G; B) = \text{tr}_{\Delta P}^{\Delta G}(H^*(\Delta P; B))$ .*

*Proof.* As a  $k(G \times G)$ -module,  $B$  has vertex  $\Delta P$  and trivial source, thus is isomorphic to a direct summand of  $\text{Ind}_{\Delta P}^{G \times G}(k)$ . Mackey’s formula shows that  $\text{Res}_{\Delta G}^{G \times G}(B)$  is still relatively  $\Delta P$ -projective, hence lemma 5 implies the result. Alternatively, this follows from the fact that  $b = 1_B$  can be written as a relative trace of the form  $b = \text{Tr}_{\Delta P}^{\Delta G}(y)$  for some  $y \in B^{\Delta P}$ .  $\square$

**Proposition 7.** *Let  $G$  be a finite group and  $B$  be a block algebra of  $kG$ . Set  $b = 1_B$  and for every  $B$ -Brauer pair  $(Q, e)$  set  $B_{(Q, e)} = kC_G(Q)eb$ . Then  $B_{(Q, e)}$  is a direct summand of  $B$  as a  $k(C_G(Q) \times C_G(Q))\Delta Q$ -module, isomorphic to  $kC_G(Q)e$ . In particular,  $H^*(\Delta Q; B_{(Q, e)})$  is a direct summand, as a graded vector space, of  $H^*(\Delta Q; B)$ , and we have*

$$H^*(\Delta G; B) = \sum_{(Q, e)} \text{tr}_{\Delta Q}^{\Delta G}(H^*(\Delta Q; B_{(Q, e)}))$$

where in the sum  $(Q, e)$  runs over a set of representatives of the  $G$ -conjugacy classes of  $B$ -Brauer pairs.

*Proof.* The proof adapts techniques that have been used in the proof of a result of Watanabe [9, Lemma 1]. Clearly  $H^*(\Delta G; B)$  contains the right side in the displayed equation. We need to show that  $H^*(\Delta G; B)$  is contained in the right side. Since any summand of the right side of the

form  $\text{tr}_{\Delta Q}^{\Delta G}(H^*(\Delta Q; B_{(Q,e)}))$  depends only on the  $G$ -conjugacy class of  $(Q, e)$  it suffices to prove the inclusion

$$H^*(\Delta G; B) \subseteq \sum_Q \text{tr}_{\Delta Q}^{\Delta G}(H^*(\Delta Q; B_Q))$$

where  $Q$  runs over the  $p$ -subgroups of  $G$  for which  $\text{Br}_Q(b) \neq 0$ . Note that this makes sense since  $B_Q$  is a direct summand of  $B$  as a  $k\Delta Q$ -module, hence  $H^*(\Delta Q; B_Q)$  is a subspace of  $H^*(\Delta Q; B)$ , to which we then apply the transfer map  $\text{tr}_{\Delta Q}^{\Delta G}$ . Since  $H^*(\Delta G; B) = \text{tr}_{\Delta P}^{\Delta G}(H^*(\Delta P; B))$  by lemma 6 it suffices to show that the right side contains  $\text{tr}_{\Delta R}^{\Delta G}(H^*(\Delta R; B))$  for any  $p$ -subgroup  $R$  of  $G$ . This will be shown by induction. For  $R = \{1\}$  this holds trivially because  $B_{\{1\}} = B$  and  $C_{\{1\}} = \{0\}$ . For  $R \neq \{1\}$  we have a direct sum decomposition  $B = B_R \oplus C_R$  of  $kN_{G \times G}(\Delta R)$ -modules as in proposition 3, and hence

$$H^*(\Delta R; B) = H^*(\Delta R; B_R) + H^*(\Delta R; C_R)$$

Since  $C_R(\Delta R) = \{0\}$  we have

$$H^*(\Delta R; C_R) \subseteq \sum_{S; S < R} \text{tr}_{\Delta S}^{\Delta R}(H^*(\Delta S; B))$$

by lemma 5. Applying the transfer map  $\text{tr}_{\Delta R}^{\Delta G}$  yields

$$\text{tr}_{\Delta R}^{\Delta G}(H^*(\Delta R; C_R)) \subseteq \sum_{S; S < R} \text{tr}_{\Delta S}^{\Delta G}(H^*(\Delta S; B))$$

hence

$$\text{tr}_{\Delta R}^{\Delta G}(H^*(\Delta R; B)) \subseteq \text{tr}_{\Delta R}^{\Delta G}(H^*(\Delta R; B_R)) + \sum_{S; S < R} \text{tr}_{\Delta S}^{\Delta G}(H^*(\Delta S; B))$$

The result follows by induction.  $\square$

**Lemma 8.** *Let  $G$  be a finite group and  $B$  be a block algebra of  $kG$ . Set  $b = 1_B$  and for every  $B$ -Brauer pair  $(Q, e)$  set  $B_{(Q,e)} = kC_G(Q)eb$ . For any integer  $n \geq 0$  we have*

$$\dim_k(\text{tr}_{\Delta Q}^{\Delta G}(H^n(\Delta Q; B_{(Q,e)}))) \leq \dim_k(H^n(\Delta QC_G(Q); kQC_G(Q)e))$$

*Proof.* Clearly  $\dim_k(\text{tr}_{\Delta Q}^{\Delta G}(H^n(\Delta Q; B_{(Q,e)}))) \leq \dim_k(\text{tr}_{\Delta Q}^{\Delta QC_G(Q)}(H^n(\Delta QC_G(Q); B_{(Q,e)})))$ . Moreover, since  $B_{(Q,e)} \cong kC_Q(Q)e$  is isomorphic to a direct summand of  $kQC_G(Q)e$ , the lemma follows.  $\square$

**Lemma 9.** *Let  $G$  be a finite group,  $B$  a block of  $kG$  and  $Z$  a subgroup of order  $p$  of  $Z(G)$ . Set  $\bar{G} = G/Z$  and denote by  $\bar{B}$  the image of  $B$  in  $k\bar{G}$  under the canonical algebra homomorphism  $kG \rightarrow k\bar{G}$ . For any integer  $n \geq 0$  we have*

$$\dim_k(H^n(\Delta G; B)) \leq p \cdot \sum_{i=0}^n \dim_k(H^i(\Delta \bar{G}; \bar{B}))$$

*Proof.* The Lyndon-Hochschild-Serre spectral sequence associated with  $G$ ,  $Z$ ,  $\bar{G}$  and  $B$  endowed with the conjugation action of  $G$  reads

$$H^i(\Delta\bar{G}; H^j(\Delta Z; B)) \Rightarrow H^{i+j}(\Delta G; B)$$

Since  $\Delta Z$  acts trivially on  $kG$ , hence on  $B$ , we have  $H^j(\Delta Z; B) \cong H^j(\Delta Z; k) \otimes_k B \cong B$ , where the last isomorphism uses that we have  $H^j(\Delta Z; k) \cong k$  because  $Z$  is cyclic. Thus  $H^n(\Delta G; B)$  is filtered by subquotients of  $H^i(\Delta\bar{G}; B)$ , with  $0 \leq i \leq n$ ; in particular,  $\dim_k(H^n(\Delta G; B)) \leq \sum_{i=0}^n \dim_k(H^i(\Delta\bar{G}; B))$ . Let  $z$  be a generator of  $Z$ . As a  $k\Delta\bar{G}$ -module,  $B$  has a filtration of the form

$$B \supseteq B(1-z) \supseteq B(1-z)^2 \supseteq \cdots \supseteq B(1-z)^{p-1} \supseteq \{0\}$$

and since  $B$  is projective as a right  $kZ$ -module, the quotient of any two consecutive terms in this filtration is isomorphic to  $\bar{B}$ . Thus the appropriate long exact sequences in cohomology imply that  $\dim_k(H^i(\Delta\bar{G}; B)) \leq p \cdot \dim_k(H^i(\Delta\bar{G}; \bar{B}))$ , whence the result.  $\square$

*Proof of Theorem 1.* Let  $f$  be the function defined in remark 2. Note that  $f(n, d) \geq f(n, d-1)$  for all  $n \geq 0$  and all  $d > 0$ . Denote by  $c(d)$  the maximum of the numbers of subgroups in finite groups of order  $p^d$ . As mentioned before, theorem 1 holds for  $n = 0$ . Clearly theorem 1 holds for  $d = 0$  because a defect zero block is a matrix algebra. Let  $n$  and  $d$  be positive integers. Then  $\text{tr}_{\Delta 1}^{\Delta G}(H^n(1; B)) = \{0\}$ . Thus, by proposition 7 and lemma 8 we have  $\dim_k(HH^n(B)) \leq \sum_{(Q,e)} \dim_k(HH^n(QC_G(Q)e))$  where in the sum  $(Q, e)$  runs over a set of representatives of the  $G$ -conjugacy classes of non-trivial  $B$ -Brauer pairs. Any such pair  $(Q, e)$  has a conjugate with  $Q$  contained in a fixed defect group  $P$ , and hence the number of summands in this sum is at most  $c(d)$ . Moreover,  $Z(QC_G(Q))$  contains  $Z(Q)$ , and hence  $QC_G(Q)$  has a non-trivial central subgroup  $Z_Q$  of order  $p$ . After replacing  $(Q, e)$  by a suitable  $G$ -conjugate, we may assume that  $QC_P(Q)$  is a defect group of  $e$  viewed as a block of  $kQC_G(Q)$ ; in particular the defect groups of  $e$  have order at most  $|P| = p^d$ . Thus the defect groups of the image  $\bar{e}$  of  $e$  in  $kQC_G(Q)/Z_Q$  have order at most  $|P|/p = p^{d-1}$ , hence  $\dim_k(HH^n(kQC_G(Q)/Z_Q\bar{e})) \leq f(n, d-1)$ . It follows from lemma 9 that  $\dim_k(HH^n(kQC_G(Q)e)) \leq p \cdot \sum_{i=0}^n f(i, d-1)$ . Together with the above remarks we get the inequality  $\dim_k(HH^n(B)) \leq p \cdot c(d) \cdot \sum_{i=0}^n f(i, d-1) = f(n, d)$ , as required.  $\square$

**Remark 10.** The strong version of Donovan's conjecture states that for a fixed integer  $d \geq 0$  there should be only finitely many Morita equivalence classes of blocks with defect at most  $d$ . If true, this would imply that there are only finitely many isomorphism classes of Hochschild cohomology algebras of blocks with defect at most  $d$ ; this remains an open problem.

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